THEOREM 8

The Invertible Matrix Theorem

Let A be a square $n \times n$ matrix. Then the following statements are equivalent. That is, for a given A, the statements are either all true or all false.

a. A is an invertible matrix.
b. A is row equivalent to the $n \times n$ identity matrix.
c. A has n pivot positions.
d. The equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.
e. The columns of A form a linearly independent set.
f. The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one.
g. The equation $A\mathbf{x} = \mathbf{b}$ has at least one solution for each $\mathbf{b}$ in $\mathbb{R}^n$.
h. The columns of A span $\mathbb{R}^n$.
i. The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ maps $\mathbb{R}^n$ onto $\mathbb{R}^n$.
j. There is an $n \times n$ matrix C such that $CA = I$.
k. There is an $n \times n$ matrix D such that $AD = I$.
l. A^T is an invertible matrix.

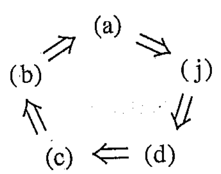
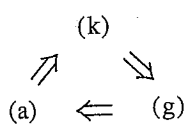


FIGURE 1

First, we need some notation. If the truth of statement (a) always implies t statement (j) is true, we say that (a) *implies* (j) and write (a) $\Rightarrow$ (j). The proof w establish the "circle" of implications shown in Fig. 1. If any one of these five stateme is true, then so are the others. Finally, the proof will link the remaining statements the theorem to the statements in this circle.

PROOF If statement (a) is true, then A^{-1} works for C in (j), so (a) $\Rightarrow$ (j). Next, (j) $\Rightarrow$ by Exercise 23 in Section 2.1. (Turn back and read the exercise.) Also, (d) $\Rightarrow$ (c) Exercise 23 in Section 2.2. If A is square and has n pivot positions, then the piv must lie on the main diagonal, in which case the reduced echelon form of A is I_n. T (c) $\Rightarrow$ (b). Also, (b) $\Rightarrow$ (a) by Theorem 7 in Section 2.2. This completes the circl Fig. 1.



(g) $\Leftrightarrow$ (h) $\Leftrightarrow$ (i)

(d) $\Leftrightarrow$ (e) $\Leftrightarrow$ (f)

(a) $\Leftrightarrow$ (l)

Next, (a) $\Rightarrow$ (k) because A^{-1} works for D. Also, (k) $\Rightarrow$ (g) by Exercise 26 in S tion 2.1, and (g) $\Rightarrow$ (a) by Exercise 24 in Section 2.2. So (k) and (g) are linke the circle. Further, (g), (h), and (i) are equivalent for any matrix, by Theorem Section 1.4 and Theorem 12(a) in Section 1.9. Thus, (h) and (i) are linked through to the circle.

Since (d) is linked to the circle, so are (e) and (f), because (d), (e), and (f) all equivalent for *any* matrix A. (See Section 1.7 and Theorem 12(b) in Section Finally, (a) $\Rightarrow$ (l) by Theorem 6(c) in Section 2.2, and (l) $\Rightarrow$ (a) by the same theo with A and A^T interchanged. This completes the proof.

Because of Theorem 5 in Section 2.2, statement (g) in Theorem 8 could als written as "The equation $A\mathbf{x} = \mathbf{b}$ has a *unique* solution for each $\mathbf{b}$ in $\mathbb{R}^n$." This stater

Section 2.3

Proofs of parts of thm 8

(k) $\Rightarrow$ (g), (j) $\Rightarrow$ (d), and

AB invertible $\Longleftrightarrow$ A and B are invertible.

$\boxed{(k) \Rightarrow (g)}$

There is an $n\times n$ matrix D such that $AD = I_n$.

$\Longrightarrow$ $A\vec{x} = \vec{b}$ is consistent for any $\vec{b} \in \mathbb{R}^n$.

Proof.- If $AD = I_n$ $\Rightarrow$ for any $\vec{b} \in \mathbb{R}^n$

$A(D\vec{b}) = (AD)\vec{b} = I_n \vec{b} = \vec{b}$. Thus, $D\vec{b}$ is a soln. of $A\vec{x} = \vec{b}$. ✓

$\boxed{(j) \Rightarrow (d)}$

There is $C_{n\times n}$ Such that $CA = I_n$

$\Rightarrow$ $A\vec{x} = \vec{0}$ has only the trivial Soln.

Proof.- Assume $\vec{x}$ is a Soln. Of $A\vec{x} = \vec{0}$.

then $\vec{x} = I_n\vec{x} = (CA)\vec{x} = C(A\vec{x}) = C\vec{0} = \vec{0}$. ✓

Key Lemma in many proofs of Square matrices

Lemma.- The reduced row echelon form of $A_{n\times n}$ has either a row of zeros or is I_n.

Proof.-

Consider all the possibilities for $A_{3\times3}$ reduced row ech. forms

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & * & 0 \\ 0 & * & 1 \\ 0 & * & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & * & * \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & * \\ 0 & 1 & * \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & * \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Clearly, if A does not have a row of zeros then, there is a pivot position in every row. Therefore, there are n pivot positions and the only possibility to have n pivot positions in n rows for a square matrix $A_{n\times n}$ is if they are along the main diagonal of A.

$\boxed{(g) \Rightarrow (b)}$ $A_{n\times n}$

$$\underbrace{A\vec{x}=\vec{b} \text{ is consistent for any } \vec{b}\in\mathbb{R}^n}_{\Downarrow} \Rightarrow A\sim I_n.$$

Proof.- The coefficient matrix A does not have a (thm 2 CHAPTER 1) row of zeros. Thus, using the previous lemma $A\sim I_n$.

$\boxed{(b) \Rightarrow (a)}$ (thm 7)

$A\sim I_n \Rightarrow A$ invertible

Proof.- $A\sim I_n \Rightarrow$ there are $E_1,..,E_r$ elementary matrices such that

$$E_r\cdots E_2E_1A = I_n$$

Since each elementary matrices $E_1,..,E_r$ are invertible we can use their inverses to obtain

$$(E_1^{-1}\cdots E_{r-1}^{-1}E_r^{-1})(E_rE_{r-1}\cdots E_1A) = (E_1^{-1}\cdots E_r^{-1})I_n$$

or

$$(E_1^{-1}\cdots E_{r-1}^{-1})(\underbrace{(E_r^{-1}E_r)}_{=I}E_{r-1}\cdots E_1A) = E_1^{-1}\cdots E_r^{-1}$$

or

$$(E_1^{-1}\cdots E_{r-1}^{-1})(E_{r-1}\cdots E_1A) = E_1^{-1}\cdots E_r^{-1}$$

After $r-1$ steps

$$(E_1^{-1}E_1)A = E_1^{-1}\cdots E_r^{-1} \Rightarrow A = E_1^{-1}\cdots E_r^{-1}$$

Therefore, A is invertible because is the product of inv. matrices.

$(a) \Rightarrow (k)$

A is invertible $\Rightarrow$ there exists D such that $AD = I_n$.

Proof.- It's enough to define $D \equiv A^{-1}$, then

$$AD = AA^{-1} = I. \checkmark$$

The sequence of proofs can be summarized as

$$k \Rightarrow g \Rightarrow b \Rightarrow a \Rightarrow k.$$

or $(a) \Leftrightarrow (k)$

A invertible $\Leftrightarrow$ there exists D such that $AD = I$

This statement can be used as an alternative definition for a square matrix A to be invertible..

Also, for practical purpose, it's convenient to write part of the above statement as

$$\boxed{\text{If there exists } D \text{ such that } AD = I \Rightarrow A \text{ is invertible and } D = A^{-1}.}$$

Proof.- Once we perform the sequence of proofs

$$k \Rightarrow g \Rightarrow b \Rightarrow a$$

We arrive to A is invertible. Therefore, there exists A^{-1} such that $A^{-1}A = AA^{-1} = I_n$, Now, $AD = I_n$ by Hypothesis

Therefore, $A^{-1}(AD) = A^{-1}I_n$ or $D = ID = (A^{-1}A)D = A^{-1}I_n = A^{-1}$ ✓

$\boxed{(d) \Rightarrow (b)}$

$\underbrace{A\vec{x} = \vec{0} \text{ has only the trivial solution}} \Rightarrow A \sim I_n$

Proof:

$\Downarrow$

A cannot have a row of zeros. otherwise,

it would be at least one free variable. As a consequence,

$A\vec{x} = \vec{0}$ would have infinitely many solutions.

Thus, $A \sim I_n$ using the key lemma.

$\boxed{(b) \Rightarrow (a)}$ $A \sim I_n \Rightarrow A$ invertible.

Proof: In previous page 3.

$A_{n\times x}$ and $B_{n\times n}$

AB invertible $\Longleftrightarrow$ A and B are invertibles.

Proof.— ($\Leftarrow$) If A and B are invertibles then,

A^{-1} and B^{-1} exists such that $AA^{-1} = I = A^{-1}A$
and $BB^{-1} = I = B^{-1}B$.

Now,
$$(AB)(B^{-1}A^{-1}) = A((BB^{-1})A^{-1}) = A(I_nA^{-1}) =$$
$$= AA^{-1} = I_n$$

Also,
$$(B^{-1}A^{-1})(AB) = B^{-1}((A^{-1}A)B) = B^{-1}(I_nB) = B^{-1}B = I_n$$

Therefore, AB is invertible and $B^{-1}A^{-1}$ is its inverse.

($\rightarrow$) AB invertible $\Rightarrow$ there exists C such that

$$(AB)C = I \quad \text{or} \quad A(BC) = I$$

Using the invertible matrix thm (IMT), we conclude that

A is invertible

Similarly, C also satisfies

$$C(AB) = I \quad \text{or} \quad (CA)B = I$$

$\overset{I.M.T}{\Longrightarrow}$ B is invertible.